

Note on the semi-continuity of the algebraic dimension.

Daniel Barlet.*

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ABSTRACT. In this short Note we show that the direct image sheaf $R^1\pi_*(\mathcal{O}_{\mathcal{X}})$ associated to an analytic family of compact complex manifolds $\pi : \mathcal{X} \rightarrow S$ parametrized by a reduced complex space S is a locally free (coherent) sheaf of \mathcal{O}_S -modules. This result allows to improve a semi-continuity type result for the algebraic dimension of compact complex manifolds in an analytic family given in [B.15].

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*Institut Elie Cartan, Géométrie,
Université de Lorraine, CNRS UMR 7502 and Institut Universitaire de France.

1 Introduction

It is well known that for a compact complex manifold X of the Fujiki-Varouchas class \mathcal{C} (recall that Varouchas [V.89] shows that this is simply the class of compact complex manifolds which admit a Kähler modification) the number

$$h^{0,1}(X) := \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X)$$

is a topological invariant (half of the first Betti number), so it is constant in an analytic deformation inside the class \mathcal{C} . In this short Note we prove that this number is invariant in any analytic deformation for any compact complex manifold. As an application, this allows us to improve a semi-continuity type result for the algebraic dimension of compact complex manifolds in an analytic family given in [B.15]. See the theorem 3.0.7 and its corollaries in section 3.

2 The result

Theorem 2.0.1 *Let $\pi : \mathcal{X} \rightarrow S$ be a holomorphic family of compact complex connected manifolds of dimension n parametrized by an irreducible complex space S . Then the coherent sheaf $R^1\pi_*(\mathcal{O}_{\mathcal{X}})$ on S is a locally free sheaf.*

The proof will use several lemmata.

Lemma 2.0.2 *Let X be a compact normal connected complex space of dimension n and let L be a holomorphic line bundle on X . Then if L and L^* have a non trivial holomorphic section, the line bundle L is holomorphically trivial.*

PROOF. Let σ and τ the non trivial holomorphic sections of L and L^* . Then the function $x \mapsto \langle \sigma(x), \tau(x) \rangle$ is holomorphic and not identically zero. So it is a non zero constant function and we see that σ and τ cannot vanish. Now the map $L \rightarrow X \times \mathbb{C}$ given by $\xi \mapsto (\pi(\xi), \xi/\sigma(\pi(\xi)))$ is holomorphic and linear on fibres with inverse the holomorphic map given by $(x, \lambda) \mapsto \lambda \cdot \sigma(x)$. So L is trivial. ■

We shall use this lemma in order to get the fact that if a holomorphic line bundle on a compact complex connected manifold is not holomorphically trivial, then L or L^* has no non trivial holomorphic section.

Proposition 2.0.3 *Let M be a reduced complex space and $(X_s)_{s \in S}$ an analytic family of compact n -cycles in M parametrized by a closed irreducible complex subset S of $\mathcal{C}_n(M)$ the space of compact n -cycles in M . Assume that each cycle in this family is reduced, normal and connected. Let \mathcal{L} be a holomorphic line bundle on M . Then the subset Σ of points in S such that the restriction $\mathcal{L}|_{X_s}$ is not holomorphically trivial is an open subset in S .*

Let $f : \mathcal{L} \rightarrow M$ be the projection. Then the direct image of compact n -cycles $f_* : \mathcal{C}_n(\mathcal{L}) \rightarrow \mathcal{C}_n(M)$ is a holomorphic map (see [B-M 1] chapter IV). We can restrict this map to the subset $Z \subset \mathcal{C}_n(\mathcal{L})$ defined by the condition that the cycles are connected and that the direct image by f of a cycle C in Z is a cycle X_s for some $s \in S$. These two conditions are analytic and closed thanks to the theorem IV 7.2.1 of [B-M 1] and to the assumption that S is a closed analytic subset in $\mathcal{C}_n(M)$.

Remark that, as we assume that X_s is normal and connected, a compact connected n -cycle C in \mathcal{L} with direct image X_s by the projection f is a section of the line bundle $\mathcal{L}|_{X_s}$.

Note that we have a closed embedding $j : S \rightarrow Z$ which associates to $s \in S$ the reduced n -cycle in Z equal to the zero section of the line bundle $\mathcal{L}|_{X_s}$.

Now if the line bundle $\mathcal{L}|_{X_s}$ has a non trivial holomorphic section the cycle $j(s)$ can move in $Z \cap f_*^{-1}(j(s))$ by homotheties in an analytic 1-dimensional family containing $j(s)$. So we have

$$\dim_{j(s)} (Z \cap f_*^{-1}(j(s))) \geq 1.$$

But the subset W of points w in Z such the inequality $\dim_w [Z \cap f_*^{-1}(f_*(w))] \geq 1$ is a closed analytic subset in Z . So the subset $\Sigma_0 := j^{-1}(W)$ is a closed analytic subset in S . Then the complement of Σ_0 is an open set in S . So if $L|_{X_0}$ has no non trivial holomorphic section, for s in this open set, $L|_{X_s}$ is not holomorphically trivial. If $L|_{X_0}^*$ has no non trivial holomorphic section we obtain in the same way an open set around 0 such that, for any s in it, $L|_{X_s}$ is not holomorphically trivial. The case when L and L^* have both a non trivial holomorphic section is excluded by the lemma 2.0.2. ■

Lemma 2.0.4 *Let $\pi : \mathcal{X} \rightarrow \Delta$ a proper holomorphic submersion of a complex manifold \mathcal{X} onto an open disc Δ with center 0 in \mathbb{C} , with n -dimensional connected fibres. Let L be a line bundle on \mathcal{X} and assume that L is holomorphically trivial on each $X_s, \forall s \in \Delta$. Then L is trivial on \mathcal{X} .*

PROOF. Consider the following data : an open disc Δ_1 in Δ , an open set \mathcal{U} in $\pi^{-1}(\Delta_1)$, a holomorphic trivialization $t : L|_{\mathcal{U}} \rightarrow \mathcal{U} \times \mathbb{C}$ and a holomorphic section $\gamma : \Delta_1 \rightarrow \mathcal{U}$ of π . Of course, choosing first a local trivialization of L on an open set in \mathcal{X} we can find such data with any point in Δ as the center of the (small) disc Δ_1 . Let $Z \subset \mathcal{C}_n(L)$ the analytic subset of connected compact n -cycles C in L such the direct image cycle $f_*(C)$ of C by the projection $f : L \rightarrow \mathcal{X}$ is one of the fibres of π . So we have a holomorphic map $g : Z \rightarrow \Delta$ defined by $f_*(C) = X_{g(C)}$. Denote now by Z_1 the subset in Z of cycles $C \in g^{-1}(\Delta_1)$ such that the cycle $t_*(C \cap f^{-1}(\mathcal{U}))$ contains the point $(\gamma(g(C)), 1) \in \mathcal{U} \times \mathbb{C}$. we want to prove the following assertions :

- 1) The subset Z_1 is a closed analytic subset of the open set $g^{-1}(\Delta_1) \subset Z$.
- 2) The projection on $L|_{\pi^{-1}(\Delta_1)}$ of the graph $\Gamma_1 \subset Z_1 \times L|_{\pi^{-1}(\Delta_1)}$ of the analytic family of compact n -cycles in L parametrized by Z_1 is a closed embedding of a

complex sub-manifold in $L|_{\pi^{-1}(\Delta_1)}$ which is disjoint of the zero section and gives a holomorphic section of $L|_{\pi^{-1}(\Delta_1)}$.

As a consequence, we shall obtain that $L|_{\pi^{-1}(\Delta_1)}$ is trivial on $\pi^{-1}(\Delta_1)$. And, as this is true for any given point s_1 in Δ and a small enough open disc Δ_1 with center s_1 , the conclusion will follow because $H^1(\Delta, \mathcal{O}_\Delta^*) = \{1\}$.

Let us prove the assertion 1). As the condition for $C \in g^{-1}(\Delta_1)$ to be in Z_1 is given by the fact that the point $(C, (\gamma(g(C)), 1))$ is in the image of the graph $\Gamma_1 \cap (Z_1 \times L|_{\mathcal{U}})$ by the proper embedding $\text{id}_{Z_1} \times t$, this is clearly a closed analytic condition as g, γ and t are holomorphic.

To prove the assertion 2), remark first that each $C \in Z_1$ is the image of a holomorphic section of $L|_{X_{g(C)}}$ which does not vanishes at the point $\gamma(g(C))$. As $L|_{X_{g(C)}}$ is trivial, this section never vanishes on $X_{g(C)}$. Remark also that g is injective in Z_1 because if $g(C) = g(C') := s$ then C and C' in Z_1 are the images of two holomorphic sections of the trivial line bundle $L|_{X_s}$ and take the same value at the point $\gamma(s)$. So $C = C'$ and $g : Z_1 \rightarrow \Delta_1$ is an isomorphism. So the analytic family of compact cycle $(C)_{C \in Z_1}$ gives exactly one holomorphic never vanishing section of $L|_{X_s}$ for each $s \in \Delta_1$. This is enough to prove our second assertion as the graph of this analytic family is a closed analytic subset in $L|_{\pi^{-1}(\Delta_1)}$ disjoint from the zero section and which is one to one on $\pi^{-1}(\Delta_1)$ by the projection of L on \mathcal{X} . \blacksquare

Lemma 2.0.5 *Let $\pi : \mathcal{X} \rightarrow \Delta$ a proper holomorphic submersion of a complex manifold \mathcal{X} onto an open disc Δ with center 0 in \mathbb{C} , with n -dimensional connected fibres. Consider the injection of sheaves on Δ*

$$j : R^1\pi_*\mathbb{Z} \rightarrow R^1\pi_*\mathcal{O}_{\mathcal{X}}.$$

The following properties are equivalent:

- 1) *Any section σ with support $\{0\}$ of the sheaf $R^1\pi_*\mathcal{O}_{\mathcal{X}}$ vanishes.*
- 2) *Any section σ of the sheaf $R^1\pi_*\mathcal{O}_{\mathcal{X}}$ such its restriction to Δ^* is in the image of the map $j : H^0(\Delta^*, R^1\pi_*\mathbb{Z}) \rightarrow H^0(\Delta^*, R^1\pi_*\mathcal{O}_{\mathcal{X}})$ is also in the image of the map $j : H^0(\Delta, R^1\pi_*\mathbb{Z}) \rightarrow H^0(\Delta, R^1\pi_*\mathcal{O}_{\mathcal{X}})$.*
- 3) *Any topologically trivial line bundle on \mathcal{X} which induces on X_0 a line bundle which is holomorphically trivial on each X_s for any $s \neq 0$ near-by enough 0 induces a line bundle which is holomorphically trivial on X_0 .*

PROOF. 1) \Rightarrow 2). Take any section σ of the sheaf $R^1\pi_*\mathcal{O}_{\mathcal{X}}$ such its restriction to Δ^* is in the image of $j : H^0(\Delta^*, R^1\pi_*\mathbb{Z}) \rightarrow H^0(\Delta^*, R^1\pi_*\mathcal{O}_{\mathcal{X}})$. As $R^1\pi_*\mathbb{Z}$ is a constant sheaf on Δ we have $H^0(\Delta^*, R^1\pi_*\mathbb{Z}) = H^0(\Delta, R^1\pi_*\mathbb{Z})$. So there exists

$\tau \in H^0(\Delta, R^1\pi_*\mathbb{Z})$ such that $\sigma - j(\tau)$ vanishes on Δ^* . Then by 1) we have $\sigma = j(\tau)$.

2) \Rightarrow 3). As Δ is Stein and contractible and we know that for each $i \geq 0$ the sheaves $R^i\pi_*\mathcal{O}_{\mathcal{X}}$ are coherent and the sheaves $R^i\pi_*(\mathbb{Z})$ are constant sheaves, the Leray spectral sequence gives natural isomorphisms $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq H^0(\Delta, R^i\pi_*\mathcal{O}_{\mathcal{X}})$ and $H^i(\mathcal{X}, \mathbb{Z}) \simeq H^0(\Delta, R^i\pi_*(\mathbb{Z}))$ for each $i \geq 0$. Then we have:

$$Cokerj := H^0(\Delta, R^1\pi_*\mathcal{O}_{\mathcal{X}})/j(H^0(\Delta, R^1\pi_*\mathbb{Z})) \simeq H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/H^1(\mathcal{X}, \mathbb{Z})$$

which classifies the holomorphic line bundles on \mathcal{X} which are topologically trivial, up to isomorphism. So the isomorphism class of a given topologically trivial line bundle L is defined by the image in $Cokerj$ of some $\sigma \in H^0(\Delta, R^1\pi_*\mathcal{O}_{\mathcal{X}})$.

Now take a line bundle L on \mathcal{X} which is topologically trivial. Assume that L is holomorphically trivial on each X_s for $s \in \Delta^*$. So, thanks to the lemma 2.0.4, this implies that the section σ corresponding to the isomorphism class of L is such that $\sigma|_{\Delta^*}$ is in $j(H^0(\Delta^*, R^1\pi_*\mathbb{Z}))$. So by 2) we obtain that σ gives 0 in $Cokerj$ and then the line bundle L is holomorphically trivial. So the restriction to X_0 is holomorphically trivial and 3) is proved.

3) \Rightarrow 1). If $\sigma \in H^0(\Delta, R^1\pi_*\mathcal{O}_{\mathcal{X}})$ vanishes on Δ^* this implies that the corresponding line bundle on \mathcal{X} is trivial on each X_s , $\forall s \in \Delta^*$. If L_{X_0} is also trivial, then the lemma 2.0.4 implies that L is trivial on \mathcal{X} . So there exists some $\tau \in H^0(\Delta, R^1\pi_*\mathbb{Z})$ such that $\sigma = j(\tau)$. But as j is injective and as $R^1\pi_*\mathbb{Z}$ is a constant sheaf, we have $\tau = 0$ and then $\sigma = 0$. So if $\sigma \neq 0$ the restriction $L|_{X_0}$ cannot be holomorphically trivial and then 3) gives a contradiction. ■

Corollary 2.0.6 *Let $\pi : \mathcal{X} \rightarrow \Delta$ a proper holomorphic submersion of a complex manifold \mathcal{X} onto an open disc Δ with center 0 in \mathbb{C} , with n -dimensional connected fibres. Then the coherent sheaf $R^1\pi_*(\mathcal{O}_{\mathcal{X}})$ is locally free.*

PROOF. It is enough to prove that the coherent sheaf $R^1\pi_*\mathcal{O}_{\mathcal{X}}$ has no torsion so that the property 1) in the previous lemma is satisfied. But the property 3) of the previous lemma is given by the proposition 2.0.3. ■

PROOF OF THE THEOREM 2.0.1. It is enough to prove that this sheaf is S -flat. But the classical “curve test” for flatness¹ is clearly satisfied thanks to the corollary 2.0.6. ■

¹A geometric way to get this is to consider the linear space associated to this coherent sheaf : then on any curve it has constant rank by corollary 2.0.6; so it is a vector bundle and the sheaf is locally free.

3 Application

As an immediate consequence of the theorem 2.0.1 we can suppress the hypothesis on the continuity of the $h^{0,1}(s)$ in the theorem 1.0.3 of [B.15] and obtain the following semi-continuity result for the algebraic dimension.

Theorem 3.0.7 *Let $\pi : \mathcal{X} \rightarrow S$ be a holomorphic family of compact complex connected manifolds of dimension n parametrized by an irreducible complex space S . Assume that there exists a dense Zariski open set S' in S such that for each s in S' the manifold X_s satisfies the $\partial\bar{\partial}$ -lemma² and such that there exists a (smooth) relative sG-form for the family $\pi|_{S'} : \mathcal{X}|_{S'} \rightarrow S'$.*

Then if $a := \inf_{s \in S'} [a(X_s)]$ we have $a(X_s) \geq a$ for each $s \in S$. ■

REMARK. A simpler statement (see remark 3 following the theorem 1.0.3 in [B.15]) which is a special case of the previous one, is obtained by assuming that the restriction of π to $\pi^{-1}(S')$ is a weakly kähler morphism in the sense of F. Campana (see for instance [C.81]); this implies the $\partial\bar{\partial}$ -lemma assumption and the existence of a smooth relative sG-form for the restriction of π over S' . □

As it is not so easy to show that a proper map is weakly Kähler (and we need less : each fibre in S' has a sG-form and satisfies the $\partial\bar{\partial}$ -lemma is enough) let me recall the following results from [B.15]

Lemma 3.0.8 *Let $\pi : \mathcal{X} \rightarrow S$ be a proper holomorphic family of compact connected complex manifolds of dimension n parametrized by an irreducible complex space S . Assume that for a point $s_0 \in S$, the manifold $X_{s_0} := \pi^{-1}(s_0)$ has a sG-form ω_0 . Then we can find a small open neighbourhood S' of s_0 in S and a relative sG-form ω on $\pi^{-1}(S')$ inducing ω_0 on X_{s_0} .*

Theorem 3.0.9 *Let $\pi : \mathcal{X} \rightarrow S$ be a holomorphic family of compact complex connected manifolds of dimension n parametrized by an irreducible complex space S . Let s_0 in S such that the manifold X_{s_0} admits a (smooth) sG-form. Then there exists an open neighbourhood S_0 of s_0 , a countable union Σ of closed irreducible analytic subsets in S_0 with no interior point and a non negative integer a such that*

(i) *For any $s \in S_0$ we have $a(X_s) \geq a$.*

(ii) *For any $s \in S_0 \setminus \Sigma$ we have $a(X_s) = a$.*

Then the following corollaries are immediate from the theorem 3.0.7 and 3.0.9.

²See for instance [Va.86].

Corollary 3.0.10 *Let $\pi : \mathcal{X} \rightarrow S$ be a holomorphic family of compact complex connected manifolds of dimension n parametrized by an irreducible complex space S . Assume that there exists a dense Zariski open set S' in S such that for each s in S' the manifold X_s is Kähler. Then if $a := \inf_{s \in S'} [a(X_s)]$ we have $a(X_s) \geq a$ for each $s \in S$.*

Corollary 3.0.11 *Let $\pi : \mathcal{X} \rightarrow S$ be a holomorphic family of compact complex connected manifolds of dimension n parametrized by an irreducible complex space S . Assume that there exists a dense Zariski open set S' in S such that for each s in S' the manifold X_s is projective. Then for each $s \in S$ the manifold X_s is Moishezon.*

We conclude by noticing that there exists an analytic family of smooth complex compact surfaces of the class VII (not Kähler) parametrized by a disc Δ such that the central fibre has algebraic dimension 0 and all other fibres have algebraic dimension 1. See [F-P.09].

This shows that in our theorem 3.0.7 some Kähler type assumption on the general fibre X_s cannot be avoided in order that the “general” algebraic dimension gives a lower bound for the algebraic dimensions of all fibres.

4 References

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